B.Sc. (Maths) part-I

paper – II

Topic:- Integration of trigonometric functions

Type ① $\int \frac{dx}{a+b\sin x}$

Example – ① $\int \frac{dx}{4+5\sin x}$

soln. – Let $I = \int \frac{dx}{4+5\sin x}$

$$= \int \frac{dx}{4\left(\sin^2\frac{x}{2}+\cos^2\frac{x}{2}\right)+5\cdot 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{4\left(\tan^2\frac{x}{2}+1\right)+10\tan\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}}{4\tan^2\frac{x}{2}+10\tan\frac{x}{2}+4}$$

Let $\tan\frac{x}{2} = z$ then $\sec^2\frac{x}{2}\cdot\frac{1}{2}dx = dz$

$$I = \int \frac{2dz}{4z^2+10z+4} = \int \frac{dz}{2z^2+5z+2}$$

$$= \int \frac{dz}{(2z+1)(z+2)}$$

$$= \frac{1}{3}\int\left\{\frac{2}{2z+1} - \frac{1}{z+2}\right\}dz$$

$$= \frac{1}{3}\left[2\log(2z+1)\cdot\frac{1}{2} - \log(z+2)\right]$$

$$= \frac{1}{3}\log\frac{2z+1}{z+2} = \frac{1}{3}\log\left[\frac{2\tan\frac{x}{2}+1}{\tan\frac{x}{2}+2}\right]$$

~~type: $\int \frac{dx}{a\sin x + b\cos x}$~~

type :- $\int \frac{dx}{a + b\sin x + c\cos x}$

Example ② $\int \frac{dx}{2 + 3\sin x + 4\cos x}$

soln:- $I = \int \frac{dx}{2\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right) + 3\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right) + 4\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right)}$

$$= \int \frac{dx}{6\cos^2\frac{x}{2} - 2\sin^2\frac{x}{2} + 6\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{6 - 2\tan^2\frac{x}{2} + 6\tan\frac{x}{2}}$$

$$= \frac{1}{2}\int \frac{\sec^2\frac{x}{2}}{3 + 3\tan\frac{x}{2} - \tan^2\frac{x}{2}}$$

we put $\tan\frac{x}{2} = z$ so that

$\frac{1}{2}\sec^2\frac{x}{2}\,dx = dz$

$$I = \frac{1}{2}\int \frac{2\,dz}{3 + 3z - z^2} = -\int \frac{dz}{z^2 - 3z - 3}$$

$$= -\int \frac{dz}{\left(z^2 - 3z + \frac{9}{4}\right) - \left(\frac{9}{4} + 3\right)}$$

$$= -\int \frac{dz}{\left(z - \frac{3}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2}$$

We put $x-\frac{3}{2}=y$ and $a=\frac{\sqrt{21}}{2}$
so that

$$I=-\int\frac{dy}{y^2-a^2}=\int\frac{dy}{a^2-y^2}=\frac{1}{2a}\log\frac{a+y}{a-y}$$

$$=\frac{1}{2}\times\frac{2}{\sqrt{21}}\log\frac{\frac{\sqrt{21}}{2}+z-\frac{3}{2}}{\frac{\sqrt{21}}{2}-z+\frac{3}{2}}$$

$$=\frac{1}{\sqrt{21}}\log\frac{\sqrt{21}-3+2z}{\sqrt{21}+3-2z}$$

$$=\frac{1}{\sqrt{21}}\log\frac{\sqrt{21}-3+2\tan\frac{x}{2}}{\sqrt{21}+3-2\tan\frac{x}{2}}$$

Type:- $\int\frac{\alpha+\beta\cos x}{a+b\cos x}dx$, $\int\frac{\alpha+\beta\sin x}{a+b\sin x}$, $\int\frac{\alpha+\beta\sin x+\gamma\cos x}{a+b\sin x+c\cos x}$

Working rule:- Express the numerator in the integrand $=l\times$(deno.) $+m$(derivative of deno.) $+n$

where l, m, n are const. to be determined by equating the co-efficients of $\cos x$ and $\sin x$ and the const terms.

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Example. $\int \frac{\cos x \, dx}{\sin x + \cos x}$

Soln: we put = α(denom.) + β(diff. coeff. of denom)

$= \cos x = \alpha(\sin x + \cos x) + \beta(\cos x - \sin x)$

$= (\alpha - \beta)\sin x + (\alpha + \beta)\cos x$

Equating the coefficient of $\cos x$ and $\sin x$ from both side we have

$\alpha + \beta = 1, \ \alpha - \beta = 0$

$\alpha = \beta = \frac{1}{2}$

$\therefore I = \int \frac{\frac{1}{2}(\sin x + \cos x) + \frac{1}{2}(\cos x - \sin x)}{\sin x + \cos x}$

$= \int \left(\frac{1}{2} + \frac{1}{2} \frac{\cos x - \sin x}{\sin x + \cos x} \right) dx$

$= \frac{1}{2}\int dx + \frac{1}{2}\int \frac{\cos x - \sin x}{\sin x + \cos x} dx$

$= \frac{1}{2}x + \frac{1}{2}\log(\sin x + \cos x)$

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